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Number of evaluation points to improve accuracy of seismic damage evaluation using point estimate method

K. Iiyama, H. Morikawa, & S. Hirose

Tokyo Institute of Technology, Tokyo, Japan.

K. Fujita

RIKEN Advanced Institute for Computational Science, Hyogo, Japan.

T. Ichimura

The University of Tokyo, Tokyo, Japan.

ABSTRACT: The objective of this study is to investigate a practical method for damage evaluation which can be applied to large-scale seismic simulation targets over a wide area with a great number of structures. In order to realize reliable damage estimations based on numerical simulations, it is necessary not only to increase model accuracy, but also to introduce model uncertainty. Rosenblueth's point estimate is one of the useful methods for damage evaluation that considers model uncertainty and requires only a small number of calculations. The point estimate method is based on a technique for numerical integration known as the Gaussian quadrature. It recovers moments of the targeted frequency distribution from only several evaluation points. The estimation accuracy may rise with an increase in the number of evaluation points; however, the number of points and the computational load are in a trade-off relation. This study examines the influence of evaluation points on the accuracy in estimating the statistical values of seismic responses and damage probability. A case study indicates that the amount of improvement in estimation accuracy gradually decreases with increases in the evaluation points. This suggests that increases in the number of evaluation points can work effectively on damage evaluations in large-scale simulations, but only up to a limited number of increases.

1 INTRODUCTION

Among the several approaches available for damage estimation, the representative one is based on structural-damage statistics, accumulating past data on seismic hazards and applying damage curves or fragility curves. Another approach is based on a large-scale numerical simulation. Recent sophisticated computational science and technology allows for such a simulation, which can perform elaborate calculations for dealing with a vast amount of information on the seismic source, the underground structures, and the various structures. At the time of a strong earthquake, if the extent of the damage to a large area can be determined on a real-time basis with a high degree of accuracy, this will lead to prompt action against disasters. The Integrated Earthquake Simulator (IES) has been developed for this purpose (e.g., Hori and Ichimura, 2008); it includes a system for seismic response analyses which can target entire cities.

Modelling approaches always include assumptions and inaccuracies to some extent. Therefore, it is desirable that such model uncertainty be considered. On the other hand, damage evaluations that consider model uncertainty generally involve numerous calculations, in order to obtain accurate frequency distributions (FDs) of the building responses; and thus, they are inadequate for large-scale simulations. Point estimate (e.g., Rosenblueth, 1975) is one of the useful methods for overcoming this disadvantage, because it requires only a small number of calculations. Although the estimation accuracy of FDs probably rises with an increase in the number of evaluation points, the number of points and the computational load are in a trade-off relation. To examine the effect of the evaluation points on estimation accuracy, the theory of point estimates is firstly reviewed focusing on the evaluation points and the accuracy in recovering the statistical values of the original FD. Next, the statistical values of the seismic response are estimated by the point estimate (PE) method for a large number of buildings, and the changes in characteristics of the statistical values are examined with the evaluation points. Finally, the structural damage is evaluated based on the exceedance probability (EP) calculated from the cumulative distribution (CD) of the seismic responses, introducing the Gram-Charlier expansion as the probability density function (PDF) and utilizing the statistical values obtained from the PE. Employing the Monte Carlo method (MCM) as a reference, the number of evaluation points required to improve the accuracy of the seismic damage evaluation is discussed.

2 POINT ESTIMATE

2.1 Evaluation points for Rosenblueth's point estimate

Let X and Y indicate an uncertain building property and the corresponding response of the building, respectively. Then, Y is a function of X, $Y=g(X)$, where X is a random variable with a probability density function, $p(X)$. Function $g(X)$ is unknown, but can be approximated by polynomials.

Rosenblueth's point estimate approximates the low-order moments of Y using some evaluation points X_i and corresponding weights P_i, as follows:

$$E[Y^k] \cong \sum_{i=1}^{m} P_i \cdot Y_i^k = \sum_{i=1}^{m} P_i \cdot g^k(X_i) \quad \text{for } m=1,2,\ldots \tag{1}$$

where m is the number of evaluation points. P_i and X_i are selected to recover the statistics of X. For instance, in the case of a two-point estimate, P_i and X_i (i=1, 2) can be obtained from the following four equations:

$$\sum_{i=1}^{m} P_i = 1, \quad \sum_{i=1}^{m} P_i X_i = \mu_X, \quad \sum_{i=1}^{m} P_i (X_i - \mu_X)^2 = \sigma_X^2, \quad \sum_{i=1}^{m} P_i (X_i - \mu_X)^3 = \nu_X \sigma_X^3, \tag{2}$$

where μ_X, σ_X, and ν_X are the mean, the standard deviation, and the skewness of $p_x(X)$, respectively. Since Rosenblueth's explanation for X_i and P_i was not detailed and seems to be intuitive, Christian et al. (1999) explained the method by introducing the following numerical integration technique known as the Gaussian quadrature:

$$\int_{-\infty}^{\infty} g(x) p(x) dx \cong \sum_{i=1}^{m} p_i g(x_i). \tag{3}$$

They revealed that the X_i and P_i of Rosenblueth's method correspond to integration point x_i and corresponding weight p_i, respectively, of the Gaussian quadrature. With respect to the necessity for Eq. (2), as the condition for recovering the moments of Y, Miller et al.'s method is also easy to understand (Miller et al., 1983). Equation (3) has equality when the order of polynomial $g(x)$ is less than or equal to $2m$-1 and can be rewritten as

$$\int_{-\infty}^{\infty} (a_0 + a_1 x + \ldots + a_k x^k) p(x) dx = \sum_{i=1}^{m} p_i (a_0 + a_1 x_i + \ldots + a_k x_i{}^k) \quad \textit{for } k \le 2m-1. \tag{4}$$

This equation clearly satisfies all coefficients a_k if we make the moments of discrete point x_i equal the moments of the original distribution. Thus, the criterion for Eq. (4) becomes

$$E[x^k] = \sum_{i=1}^{m} p_i x_i^k \quad \textit{for } k \le 2m-1. \tag{5}$$

Since the left side of Eq. (4) is $E[g(x)]$ itself, Eqs. (4) and (5) indicate that when integration points x_i are selected to recover the moments of x, $\mu_y = E[g(x)]$ can be obtained as the expectation value using only less than or equal to $2m$-1 of the x_i and the corresponding weights p_i.

Furthermore, PE recovers the high-order moments of a polynomial, $Y=g(X)$, using m integration points of the Gaussian quadrature. This can be understood from the approach reviewed in the next section.

2.2 Number of evaluation points and accuracy of recovered moments

In order to understand the relation between the number of evaluation points and the accuracy of the recovered statistics in more detail, we review a part of Hong (1998)'s explanation. The *k*th-order central moment of *X* is

$$M_k^{'} = \int_{-\infty}^{\infty} (x - \mu_x)^k p(x) dx, \tag{6}$$

where μ_x is the mean of *x*. Let the polynomial $g(x)$ represent the Taylor expansion around μ_x as

$$g(x) = g(\mu_x) + \sum_{i=1}^{\infty} \frac{1}{i!} g^{(i)}(\mu_x)(x - \mu_x)^i . \tag{7}$$

Then, the expected value for *Y* is represented by

$$\mu_y = E[g(X)] = \int_{-\infty}^{\infty} g(x) p(x) dx = g(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} g^{(i)}(\mu_x) M_i^{'} . \tag{8}$$

Using the *i*th evaluation point, x_i, written by $x_i = \mu_x + \xi_i \sigma_x$ (*i*=1,2,…), and considering Eq. (7), the left side of Eq. (1) for $k = 1$ is given by

$$\sum_{k=1}^{m} p_k g(x_k) = g(\mu_x) \sum_{k=1}^{m} p_k + \sum_{i=1}^{\infty} \frac{1}{i!} g^{(i)}(\mu_x) \left(\sum_{k=1}^{m} p_k \xi_k^i \right) \sigma_x^i . \tag{9}$$

To approximate the exact mean value for μ_y by Eq. (9), we can match the first 2*m* terms on the right side of Eqs. (8) and (9).

$$\sum_{k=1}^{m} p_k \xi_k^i = M_i^{'} / \sigma_x^i = \lambda_{x,i} \quad \text{for } i = 0, ..., 2m - 1 . \tag{10}$$

From the above 2*m* equations for each *i*, unknown parameters p_k and ξ_k (*k*=1,...,*m*) can be obtained. Then, the first-order moment of *Y* is given from Eqs. (8) and (9) as

$$E[Y] = \mu_Y = \sum_{k=1}^{m} p_k g(x_k) + \sum_{i=2m}^{\infty} \frac{1}{i!} g^{(i)}(\mu_x)(\lambda_{x,i} - \sum_{k=1}^{m} p_k \xi_k^i) \sigma_x^i \tag{11}$$

If $g(x)$ is the (2*m*-1)th-order polynomial, $g^{(i)}(\mu_x)$ is equal to zero and PE gives an exact solution for μ_x. In addition, Eq. (11) indicates that the estimation error will increase with standard deviation σ_x for a larger-order polynomial. The second-order moment is given by

$$\begin{aligned} E[Y^2] = {} & \sum_{k=1}^{m} p_k g^2(x_k) + 2g(\mu_x) \sum_{i=2m}^{\infty} \frac{1}{i!} g^{(i)}(\mu_x)(\lambda_{x,i} - \sum_{k=1}^{m} p_k \xi_k^i) \sigma_x^i \\ & + \sum_{i}^{\infty} \sum_{j}^{\infty} \frac{1}{i!} \frac{1}{j!} g^{(i)}(\mu_x) g^{(j)}(\mu_x)(\lambda_{x,i+j} - \sum_{k=1}^{m} p_k \xi_k^{i+j}) \sigma_x^{i+j} \quad (i + j \ge 2m) . \end{aligned} \tag{12}$$

Similarly, the *n*th-order moment includes the term with the *n*th power of $g^{(i)}(\mu_x)$. Therefore, the exact solution for the *n*th-order moment can be obtained if $g(x)$ is less than the (2*m*/*n*)th-order. Thus, if the function of interest, $g(x)$, has a complex form, an increase in evaluation points X_i is effective for recovering the higher moments of *Y*. In addition, care must be taken as the estimation accuracy for the second-order central moment, $\sigma_Y^2 = E[Y^2] - \mu_Y^2$, may become less than that for the second-order moment, E[Y^2], when the estimation accuracy for μ_Y decreases.

3 DAMAGE EVALUATION USING POINT ESTIMATE

PE recovers the statistical values for *Y*, as described in a previous section. However, the information is not enough to evaluate the damage probability. Therefore, this study employs a PDF which can approximate the FD of Y using the high-order statistics obtained by PE.

This study applies the Gaussian distribution to the PDF of stochastic variable X in the numerical study. Hence, the Gaussian-related distribution function is applied as the PDF with respect to Y. There are representative PDFs, such as the normal or the lognormal distributions. However, they use only the first two-order central moments (mean and variance). To take higher-order moments into account, the Gram-Charlier expansion (GCE) is applied here. Nakajima and Morikawa (2009) proposed the use of GCE as the PDF of the building response using statistics obtained by three point estimates, and showed the applicability of GCE with PE to non-linear problems. GCE is represented by

$$p(y)=\left[1+\frac{c_3}{6\sigma^3}h_3(\frac{y-\mu_Y}{\sigma_Y})+\frac{c_4}{24\sigma^4}h_4(\frac{y-\mu_Y}{\sigma_Y})+...\right]\phi(\frac{y-\mu_Y}{\sigma_Y}) \tag{13}$$

where ϕ represents the normal probability distribution and $h_k(y)$ is the kth order Hermite polynomial which can evaluate a shift from the normal probability distribution.

$$h_k(y)=e^{\frac{y^2}{2}}\left(-\frac{d}{dy}\right)^k e^{-\frac{y^2}{2}}. \tag{14}$$

where c_n is the nth-order cumulant which can be determined from the nth-order moment, m_n, given as the expected value of function $p(y)$.

$$m_n=\int_{-\infty}^{\infty} y^n p(y)dy \tag{15}$$

From an engineering viewpoint, this study takes the fourth-order Hermite polynominal in Eq. (13), that is, only $h_3(y)$ and $h_4(y)$ are considered. Then, moment m_n, $n\leq 4$, is employed.

The EP for a value of θ for the ith building can be evaluated from the CD of $p(y)$ as follows:

$$EP_{\theta,i}=1.0-\int_{-\infty}^{\theta} p_i(y)dy \qquad (0\leq EP_{\theta,i}\leq 1) \tag{16}$$

To validate the estimation accuracy of this method using PE, the Monte Carlo method (MCM) is also employed as a reference and the results are compared.

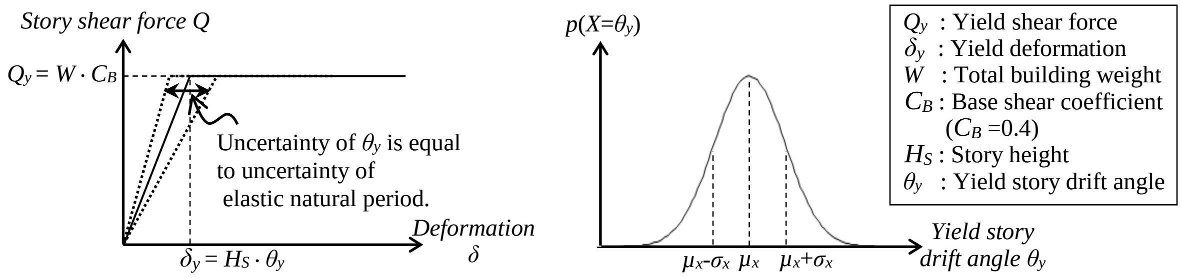
4 APPLICATION TO LARGE-SCALE SEISMIC SIMULATION

4.1 Problem settings

Using the IES code, nonlinear analyses (time interval of 0.0005 s) by the multi-degree-of-freedom (MDOF) shear spring model were performed by targeting 7,348 buildings, the number of stories (Ns) of each one being from one to 37. Inputting the El Centro earthquake to a depth of -50 m from the ground surface, the acceleration waves on the ground surface are calculated by a one-dimensional amplification analysis considering the ground structure under each building. Rayleigh damping is applied as the viscous damping matrix, considering a damping factor of 2.0 or 3.0% for the first and second modes. The damping factor is selected according to the building classification.

4.2 Model uncertainty and analytical case

Since the intention of this study is to evaluate a method for damage evaluation using PE itself, rather than a damage evaluation of a specified area, the simplest building model is used. Mass and story stiffness are set to be independent of the story, and story stiffness is modeled by a normal bilinear model. The bilinear model has strong non-linearity and it probably leads the shape of $g(X)$ to be more complex than general models, e.g., the tri-linear model. Model uncertainty is given to yield story drift angle θ_y and is set at $p(X)$ so as to follow the Gaussian distribution $\varphi(\mu_x, \sigma_x)$, as shown in Fig. 1. Analytical cases are prepared, as shown in Table 1, for each set of μ_x and coefficient of variation (COV) for θ_y. (The standard deviation is $\sigma_x=\mu_x\times$COV.) The damage to the buildings in this study is evaluated by a maximum story drift angle, θ_{max}.



(a) Stiffness model with stochastic variable (b) PDF of θ_y

Figure 1. Uncertainty of building properties

Table 1. Analytical cases

Property of θy	**Case A1**	**Case A2**	**Case A3**	**Case B1**	**Case B2**	**Case B3**	**Case C1**	**Case C2**	**Case C3**
mean μ_X	1/200	1/200	1/200	1/150	1/150	1/150	1/100	1/100	1/100
COV	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3

*COV: Coefficient of variation

4.3 Setting of evaluation points

To examine the influence of the number of evaluation points on the damage estimation, we apply 3, 5,…, 15 evaluation points to each calculation case. When $p(X=\theta_y)$ is normal distribution $\varphi(0, \sigma_X)$, Eq. (3) results in the Gauss-Hermite quadrature after an integral transformation. Then, the table of integration points x_i and the corresponding weight p_i for $i \leq 20$ for the Gauss-Hermite quadrature provided by Salzer and Capuano (1952) can be used. Since $x_i > 0$ is necessary in this study, x_i is reset to a minimum value of 0.002 if the x_i is equal to or less than zero. Hereafter, each point estimate is represented by "3PE", "5PE",…, "15PE".

4.4 Estimation of statistics and errors

As for the results for PE, the general properties of θ_{max} obtained by MCM are shown in Table 2 for each case. Here, subscript k indicates the building number (k =1, … , 7348), and subscripts *_ave* and *_std* represent the average and the standard deviations for all the buildings, respectively. For reference, the level of ductility calculated by μ_{Yk_ave}/μ_X is noted in brackets. The table indicates that θ_{max} tends to generally be "Case C > Case B > Case A". In addition, the standard deviation becomes large with COV, except for Cases B2 and B3.

Table 2. General properties of frequency distribution of θ_{max} for 7348 buildings

Properties	**Case A1**	**Case A2**	**Case A3**	**Case B1**	**Case B2**	**Case B3**	**Case C1**	**Case C2**	**Case C3**
μ_{Yk_ave}	0.020 (4.0)	0.019 (3.8)	0.019 (3.8)	0.022 (4.4)	0.022 (4.4)	0.019 (3.8)	0.028 (5.6)	0.028 (5.6)	0.027 (5.4)
μ_{Yk_std}	0.0128	0.0123	0.0119	0.0123	0.0123	0.0123	0.0147	0.0141	0.0136
σ_{Yk_ave}	0.0027	0.0035	0.0043	0.0031	0.0044	0.0035	0.0040	0.0053	0.0062
σ_{Yk_std}	0.0016	0.0019	0.0022	0.0018	0.0023	0.0019	0.0025	0.0029	0.0032

*( ): the value is the average μ_{Yk_ave} divided by μ_X

As an example of the results, the accuracy of the first three-order moments (m_1, m_2, and m_3), estimated by PE, are shown in Fig. 2. The figure presents a comparison between 3PE and 9PE for Case C1. Totally, the moments for PE approximately correspond to those for MCM, and their correlations are high. The minimum correlation coefficients among the nine cases are 0.973 and 0.995 for 3PE and 9PE, respectively, as shown in Table 3. Although the accuracy of the moment estimation for 9PE is better than that for 3PE, there seems to be no large difference.

In contrast, the accuracy of the second-order central moment σ_Y for 3PE, 7PE, and 13PE is shown in Fig. 3. As noted in section 2.2 (Eqs. (12) and (13)), the accuracy of σ_Y is not high because the small

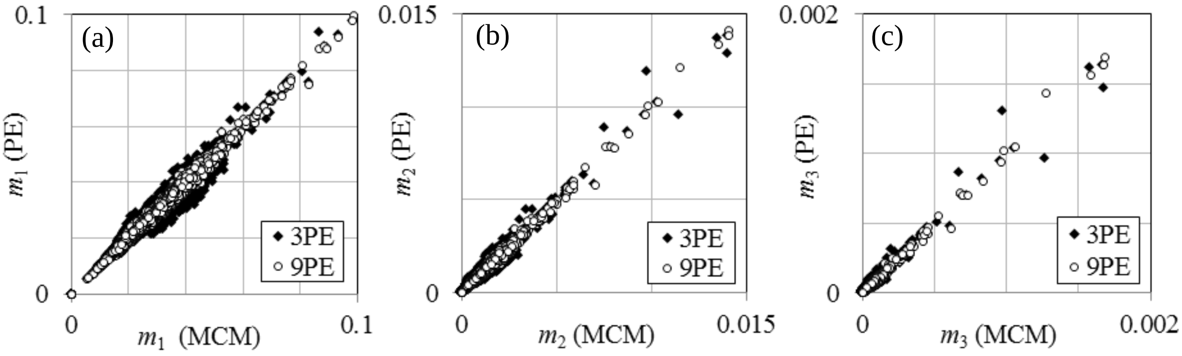


Figure 2. Comparison of estimation accuracy of moments of θ_{maxi} between two kinds of PEs (Case C1): (a) m_1, (b) m_2, and (c) m_3

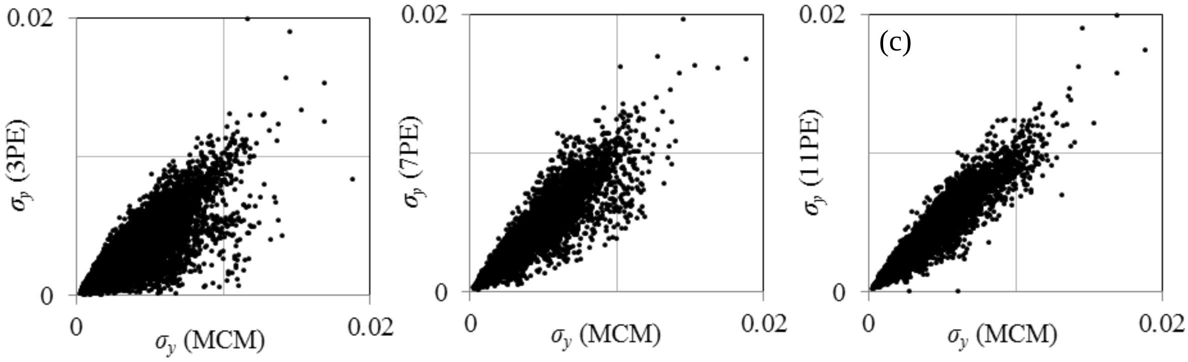


Figure 3. Estimation accuracy of second-order central moment of θ_{maxi} for three kinds of PEs (Case C1): (a) 3PE, (b) 7PE, and (c) 11PE

Table 3. Minimum correlation coefficient among nine cases

PE	m_1	m_2	m_3	m_4	σ_Y
3PE	0.973	0.995	0.994	0.991	0.731
9PE	0.995	0.999	0.999	0.999	0.869

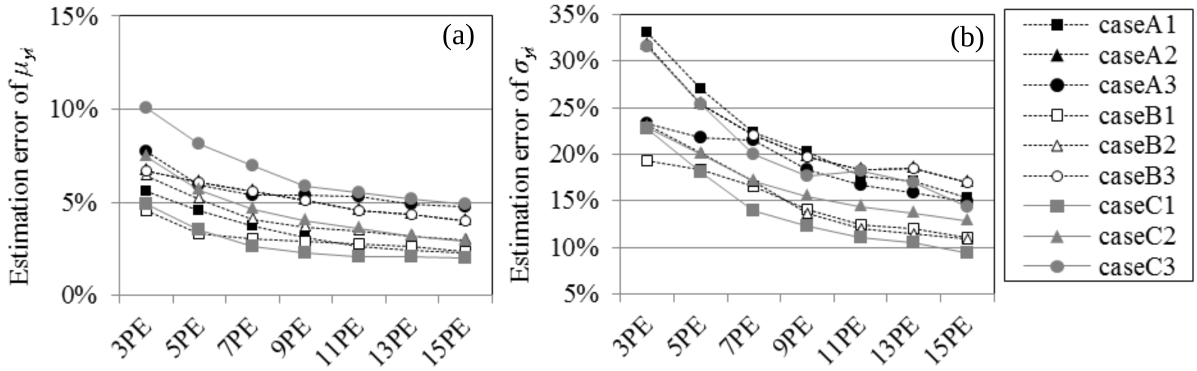


Figure 4. Error in estimation of statistics: (a) μ_{yi} and (b) σ_{yi}

estimation errors for m_1 and m_2 directly influence it. Therefore, the correlation coefficient for σ_Y becomes less than that for the moments, as shown in Table 3. Not only the moments, but also the central moments are used to estimate PDF by GC, and it is desirable that the accuracy of both be as high as possible. However, the effectiveness of the increase in evaluation points is also clear according to Fig. 3. Figure 4 shows the changes in error in the estimation of statistics μ_Y and σ_Y with the number of evaluation points. The figure indicates that with a greater number of evaluation points, the accuracy of the estimation of the statistics will be higher. In addition, the change seems to be exponential-like and approximately becomes small from 7PE or 9PE. This also indicates that the effectiveness of increasing the number of evaluation points is high around seven or nine points in this study.

4.5 Estimation of exceedance probability and accuracy

Using the statistics obtained in the previous section, $EP_{\theta,i}$, which represents a probability and exceeds a targeted maximum story drift angle θ for the i-th building, is calculated from Eq. (17) for θ =1/100 and 1/50. Hereafter, “GC3”, “GC5”,…, “GC15” represent the results obtained from the Gram-Charlier expansions for 3PE, 5PE,…, 15PE, respectively. Figures 5 and 6 show the comparison of $EP_{1/100,i}$ and $EP_{1/50,i}$ for Case A1. From these figures, the estimation accuracy of the exceedance probability greatly improves with an increase in the number of evaluation points for both $EP_{1/100,i}$ and $EP_{1/50,i}$ in this case.

The rate which indicates that the estimation error does not exceed 10% is calculated for all cases by

$$R_{\theta,err\le 10\%} = N_{\theta,err\le 10\%} / N_b \qquad (\theta = 1/100, 1/50)$$

where N_b is the number of buildings (N_b=7348) and $N_{\theta,err\le 10\%}$ is the number of i which satisfies the following inequalities for each θ.

$$EP_{\theta},_i(MCM) - 0.1 \le EP_{\theta},_i(GC) \le EP_{\theta},_i(MCM) + 0.1 \qquad for \;\; i = 1,...,Nb$$

Figure 7 presents the changes in rates $R_{1/100,err\leqq 10\%}$ and $R_{1/50,err\leqq 10\%}$ with the number of evaluation points. Most cases show that the accuracy of the damage evaluation rises with an increase in the number of evaluation points. However, the amount of improvement in accuracy also falls with an increase in the number of evaluation points. The tendency is likely to become stronger when the estimation accuracy is extremely low. These results indicate that an increase in the number of evaluation points of only several numbers will effectively bring about high estimation accuracy for the damage evaluations based on point estimates with the Gram-Charlier expansion.

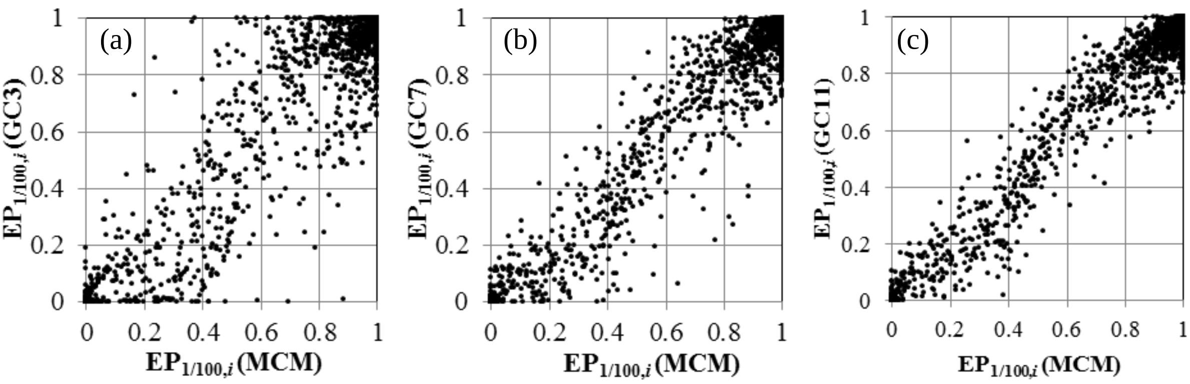


Figure 5. Comparison of exceedance probabilities $EP_{1/100,i}$ (Case A1): (a) GC3, (b) GC7, and (c) GC11

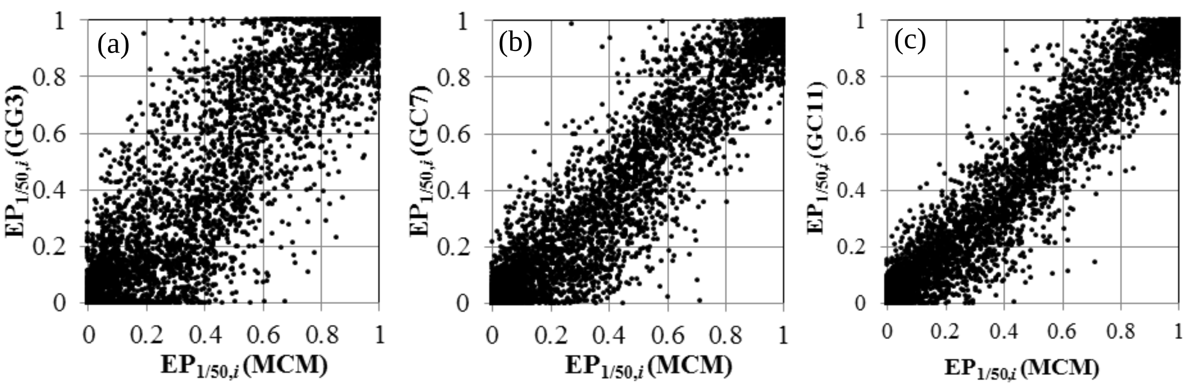


Figure 6. Comparison of exceedance probabilities $EP_{1/50,i}$ (Case A1): (a) GC3, (b) GC7, and (c) GC11

CaseA1

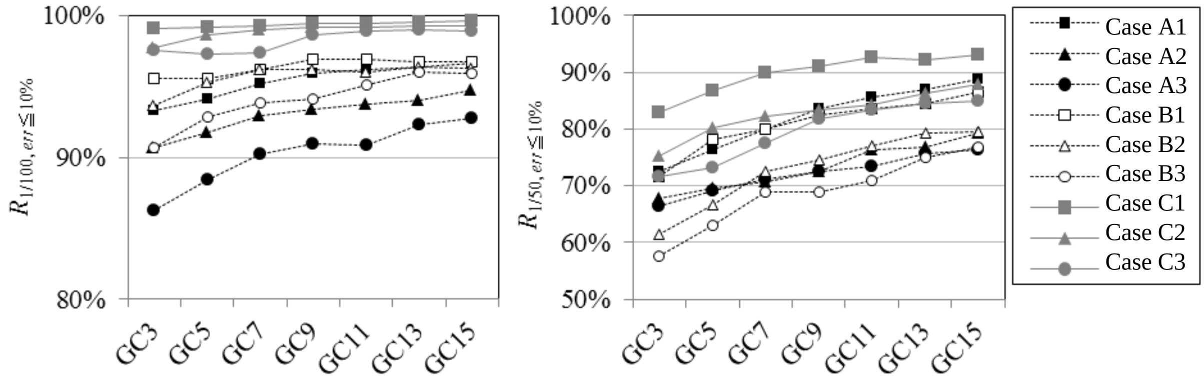


Figure 7. Rate indicates that estimation error does not exceed 10%: (a) EP1/100 and (b) EP1/50

5 CONCLUDING REMARKS

As a practical method for the damage estimation of building structures with model uncertainty, based on a large-scale numerical simulation, a damage-evaluation method, founded on point estimates using the Gram-Charlier expansion, was introduced. The influence of the evaluation points on the accuracy in estimating the statistical values of seismic responses and damage probability was examined. In the present simulation cases, the amount of improvement in estimation accuracy gradually decreased with an increase in the number of evaluation points. This suggests that a limited number of increases in evaluation points is effective for damage evaluations in large-scale simulations.

High accuracy in estimations may be realized by smaller numbers of evaluation points if a more realistic model and its uncertainty are employed. On the other hand, further studies are needed to address problems with several variables; not using so many evaluation points may still be disadvantageous. The effectiveness of the selection of evaluation points needs to be examined not only by focusing on the response properties, but also by considering the calculation load using a more realistic model and uncertainty.

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